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Abstract

We define Letac-Wesolowski-Matsumoto-Yor (LWMY) functions as decreasing functions from $(0,\infty)$ onto $(0,\infty)$ with the following property: there exist independent, positive random variables X and Y such that the variables $f(X+Y)$ and $f(X)-f(X+Y)$ are independent. We prove, under additional assumptions, that there are essentially four such functions. The first one is $f(x)=1/x$. In this case, referred to in the literature as *the Matsumoto-Yor property*, the law of X is generalized inverse Gaussian while Y is gamma-distributed. In the three other cases, the associated densities are provided. As a consequence, we obtain a new relation of convolution involving gamma distributions and Kummer distributions of type 2.

Keywords: Gamma distribution; generalized inverse Gaussian distribution; Matsumoto-Yor property; Kummer distribution.

1 Introduction

Many papers have been devoted to the generalized inverse Gaussian (GIG) distributions since their definition by Good (1953)(see for instance Barndorff-Nielsen and Halgreen (1977), Letac and Seshadri (1983), Vallois (1989), Vallois (1991)).
The GIG distribution with parameters $\mu\in\mathbb{R}$, $a,b>0$ is the probability measure :

$$GIG(\mu,a,b)(dx)=\left(\frac{b}{a}\right)^{\mu}\frac{x^{\mu-1}}{2K_\mu(ab)}e^{-\frac{1}{2}(a^2x^{-1}+b^2x)}\mathbf{1}_{(0,\infty)}(x)dx \tag{1.1}$$

where K_μ is the classical McDonald special function.

1) Let us stress the close links between GIG, gamma distributions and the function $f_0(x)=1/x$ $(x>0)$.

a) The family of GIG distributions is invariant under f_0: we can easily deduce from (1.1) that the image of GIG(μ,a,b) by f_0 is GIG$(-\mu,b,a)$.

b) Barndorff-Nielsen and Halgreen (1977) proved:

$$GIG(-\mu,a,b)\,*\,\gamma(\mu,\frac{b^2}{2})=GIG(\mu,a,b),\quad \mu,\ a,\ b>0 \tag{1.2}$$

where $\gamma(\mu, b^2/2)(dx) = \frac{b^{2\mu}}{2^\mu\Gamma(\mu)} x^{\mu-1} \exp -\frac{b^2}{2}x\, \mathbf{1}_{(0,\infty)}(x)dx$.
Therefore if $X \sim \mathrm{GIG}(-\lambda, a, a)$ and $Y \sim \gamma(\lambda, a^2/2)$ are independent r.v.'s then

$$X \stackrel{(d)}{=} f_0(X+Y). \tag{1.3}$$

Letac and Seshadri (1983) proved that (1.3) characterizes GIG distributions of the type $\mathrm{GIG}(-\lambda, a, a)$.

c) Almost sure realizations of (1.2) have been given by Bhattacharya and Waymire (1990) in the case $\mu = \frac{1}{2}$, Vallois (1991) for any $\mu > 0$ by means of a family of transient diffusions and Vallois (1989, theorem on p.446) in terms of random walks.

2) The so-called *Matsumoto-Yor property* is the following: let X and Y be two independent r.v.'s such that

$$X \sim \mathrm{GIG}(-\mu, a, b), \;\; Y \sim \gamma(\mu, b^2/2), \quad (\mu, a, b > 0). \tag{1.4}$$

Then

$$U := \frac{1}{X+Y} = f_0(X+Y), \;\; V := \frac{1}{X} - \frac{1}{X+Y} = f_0(X) - f_0(X+Y) \tag{1.5}$$

are independent and

$$U \sim \mathrm{GIG}(-\mu, b, a), \;\; V \sim \gamma(\mu, a^2/2). \tag{1.6}$$

The case $a = b$ was proved by Matsumoto and Yor (2001) and a nice interpretation of this property via Brownian motion was given by Matsumoto and Yor (2003). The case $\mu = -\frac{1}{2}$ of the Matsumoto-Yor property can be retrieved from an independence property established by Barndorff-Nielsen and Koudou (1998) (see Koudou, 2006).
Letac and Wesolowski (2000) proved that the Matsumoto-Yor property holds for any $\mu, a, b > 0$ and characterizes the GIG distributions. More precisely, consider two independent and non-Dirac positive r.v.'s X and Y such that U and V defined by (1.5) are independent, then there exist $\mu, a, b > 0$ such that (1.4) holds.

The origin of this paper is to understand the link between the function $f_0 : x \mapsto 1/x$ and the GIG distributions in the Matsumoto-Yor property.

Obviously, the Matsumoto-Yor property can be reexpressed as follows: the image of the probability measure (on $\mathbb{R}_+^2$) $\mathrm{GIG}(-\mu, a, b) \otimes \gamma(\mu, b^2/2)$ by the transformation T_{f_0} : $(x, y) \mapsto (f_0(x+y),\, f_0(x) - f_0(x+y))$ is the probability measure $\mathrm{GIG}(-\mu, b, a) \otimes \gamma(\mu, a^2/2)$. This formulation of the Matsumoto-Yor property joined with the Letac and Wesolowski result lead us to determine the triplets (μ_X, μ_Y, f) such that

a) μ_X, μ_Y are probability measures on $(0, \infty)$,

b) $f : (0, \infty) \to (0, \infty)$ is bijective and decreasing,

c) if X and Y are independent r.v.'s such that $X \sim \mu_X$ and $Y \sim \mu_Y$ then the r.v.'s $U = f(X+Y)$ and $V = f(X) - f(X+Y)$ are independent.

Unfortunately we have not been able to solve this question without restriction. Our method can be applied provided that f is smooth and μ_X and μ_Y have smooth density functions (see Theorem 3.1 for details). After long and sometimes tedious calculations we prove (cf Theorem 2.2) that there are only four classes $\mathcal{F}_1, \ldots, \mathcal{F}_4$ of functions f such that

T_f keeps the independence property. Then, for any $f \in \mathcal{F}_i$, $1 \le i \le 4$ we have been able to give the corresponding distributions of X and Y and the related laws of U and V (for $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ see Theorems 2.4, 2.14 and Remark 2.5). The first class $\mathcal{F}_1 = \{\alpha/x;\, \alpha > 0\}$ corresponds to the known case $f = f_0$. This case is dealt with in Appendix 5.1 where we recover, under stronger assumptions, the result of Letac and Wesolowski that the only possible distributions for X and Y are GIG and gamma respectively. The proof of Letac and Wesolowski is completely different from ours since the authors made use of Laplace transforms and of a characterization of the GIG laws as the distribution of a continued fraction with gamma entries. We have not been able to develop a proof as elegant as theirs. The reason is that, with $f = f_0$ we have algebraic properties (for instance continued fractions), while these properties are lost if we start with a general function f.

It is worth pointing out that one interesting feature of our analysis is an original characterization of the families of distributions $\{\beta_\alpha(a,b,c);\, a,b,\alpha > 0,, c \in \mathbb{R}\}$ and the Kummer distributions $\{K^{(2)}(a,b,c);\, a,c > 0,, b \in \mathbb{R}\}$ (see (2.14) and (2.29) respectively). The Kummer distributions appear as the law of some random continued fractions (see Marklov *et al*, 2008, p.3393 mentioning a work by Dyson (1953) in the setting of random matrices).

As by-products of our study we obtain new relations of convolution. For simplicity we only detail the case of Kummer distributions of type 2:

$$K^{(2)}(a,b,c) * \gamma(b,c) = K^{(2)}(a+b,-b,c). \tag{1.7}$$

Obviously, this relation is similar to (1.2).

Inspired by the result of Letac and Wesolowski (2000) and Theorem 2.6, one can ask, for future work, whether a characterization of Kummer distributions could be obtained by an "algebraic" method.

As recalled in the above item c), there are various almost sure realizations of (1.2) and of the convolution coming from the Matsumoto-Yor property. One interesting open question derived from our study would be to determine a r.v. Z with distribution $K^{(2)}(a+b,-b,c)$ which can be decomposed as the sum of two explicit independent r.v.'s X and Y such that $X \sim K^{(2)}(a+b,-b,c)$ and $Y \sim \gamma(b,c)$.

The paper is organized as follows. We state our main results in Section 2. In Section 3 we give a key differential equation involving f and the log densities of the independent r.v.'s X and Y such that $f(X+Y)$ and $f(X)-f(X+Y)$ are independent (cf Theorem 3.1). Based on this equation we prove (cf Theorem 3.9) that there are only four classes of such functions f. The theorems stated in Section 2 are proved in Section 4. However, one technical proof has been postponed in Appendix.

2 Main results

Definition 2.1 *Let $f:(0,\infty)\to(0,\infty)$ be a decreasing and bijective function.*

1) Associated with f let us consider the transformation

$$T_f:\ (0,\infty)^2 \to (0,\infty)^2$$

$$(x,y) \mapsto (f(x+y), f(x) - f(x+y)). \tag{2.8}$$

The transformation T_f is one-to-one and if f^{-1} is the inverse of f, then

$$(T_f)^{-1} = T_{f^{-1}}. \tag{2.9}$$

2) Let X and Y be two independent and positive random variables. Let us define

$$(U,V) = T_f(X,Y) = (f(X+Y), f(X) - f(X+Y)). \tag{2.10}$$

f is said to be a LWMY function with respect to (X,Y) if the random variables U and V are independent. f is said to be a LWMY function if it is a LWMY function with respect to some random vector (X,Y).

One aim of this paper is to characterize LWMY functions. Let us introduce

$$f_1(x) = \frac{1}{e^x - 1}, \quad x > 0, \tag{2.11}$$

$$g_1(x) = f_1^{-1}(x) = \ln\left(\frac{1+x}{x}\right), \quad x > 0 \tag{2.12}$$

and, for $\delta > 0$,

$$f_\delta^*(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right), \quad x > 0. \tag{2.13}$$

Theorem 2.2 *Let $f : (0,\infty) \to (0,\infty)$ be decreasing and bijective. Under some additional assumptions (see Theorem 3.1, (3.7) and(3.8)), f is a LWMY function if and only if, either $f(x) = \frac{\alpha}{x}$ or $f(x) = \frac{1}{\alpha} f_1(\beta x)$ or $f(x) = \frac{1}{\alpha} g_1(\beta x)$ or $f(x) = \frac{1}{\alpha} f_\delta^*(\beta x)$ for some $\alpha, \beta, \delta > 0$.*

Remark 2.3 1) The four classes of LWMY functions are $\mathcal{F}_1 = \{\alpha/x;\ \alpha > 0\}$, $\mathcal{F}_2 = \{\frac{1}{\alpha} f_1(\beta x);\ \alpha, \beta > 0\}$, $\mathcal{F}_3 = \{\frac{1}{\alpha} g_1(\beta x);\ \alpha, \beta > 0\}$, $\mathcal{F}_4 = \{\frac{1}{\alpha} f_\delta^*(\beta x);\ \alpha, \beta > 0\}$.

2) It is clear that if f is a LWMY function, then the functions f^{-1} and $x \mapsto \frac{1}{\alpha} f(\beta x)$, $\alpha, \beta > 0$ are LWMY functions.

3) The image of $\mathcal{F}_2$ by the map $f \mapsto f^{-1}$ is $\mathcal{F}_3$. The functions $x \mapsto \alpha/x$ and f_δ are involutive.

In the sequel we focus on the three new cases : either $f = f_1$ or $f = g_1$ or $f = f_\delta^*$ and in each case we determine the laws of the related random variables.

2.1 The cases $f = g_1$ and $f = f_1$

a) Recall the definition of the gamma distribution $\gamma(\lambda, c)(dx) = \frac{c^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-cx} \mathbf{1}_{(0,\infty)}(x) dx$, $(\lambda, c > 0)$ and the beta distribution $\mathrm{Beta}(a,b)(dx) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \mathbf{1}_{\{0<x<1\}} dx$, $(a, b > 0)$. Consider (see for instance Ng and Kotz, 1995, or Nagar and Gupta, 2002 and references therein) the *Kummer distribution of type 2* :

$$K^{(2)}(a,b,c) := \alpha(a,b,c) x^{a-1}(1+x)^{-a-b} e^{-cx} \mathbf{1}_{(0,\infty)}(x) dx, \ a, c > 0,\ b \in \mathbb{R} \tag{2.14}$$

where $\alpha(a,b,c)$ is a normalizing constant.

Associated with a couple (X,Y) of positive r.v.'s consider

$$(U,V) := T_{f_1}(X,Y) = \left(\frac{1}{e^{X+Y}-1}, \frac{1}{e^X-1} - \frac{1}{e^{X+Y}-1}\right). \tag{2.15}$$

In Theorems 2.4 and 2.6 below we suppose that all r.v.'s have positive and twice differentiable densities.

First we consider the case $f = f_1$. We determine the distributions of X and Y such that f_1 is a LWMY function associated to (X,Y).

Theorem 2.4 *1) Consider two positive and independent random variables X and Y. The random variables U and V defined by (2.15) are independent if and only if the densities of Y and X are respectively*

$$\begin{aligned} p_Y(y) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}} && (2.16)\\ p_X(x) &= \alpha(a+b,c,-a)e^{-(a+b)x}(1-e^{-x})^{-b-1} \\ &\quad \exp\left(-c\frac{e^{-x}}{1-e^{-x}}\right)\mathbf{1}_{\{x>0\}}. && (2.17)\end{aligned}$$

where a, b and c are constants such that $a,b,c>0$ and $\alpha(a+b,c,-a)$ is the constant of Equation (2.14). Thus the law of Y is the image of the Beta(a,b) distribution by the transformation $z \in (0,1) \mapsto -\log z \in (0,\infty)$, while the law of the variable $f_1(X)$ is $K^{(2)}(a+b,-b,c)$ (cf Equation (2.14)).

2) If 1) holds then $U \sim K^{(2)}(a,b,c)$ and $V \sim \gamma(b,c)$.

The proof of Theorem 2.4 will be given in Section 4.

Remark 2.5 Since $g_1 = f_1^{-1}$, Remark 2.3 and Theorem 2.4 imply that the r.v.'s associated with the LWMY function g_1 are the r.v.'s U and V distributed as in item 2. of Theorem 2.4.

b) As suggest identities (2.16) and (2.17) it is actually possible to simplify the statement of Theorem 2.4. Since $T_{g_1} = T_{f_1}^{-1}$, then

$$(X,Y) = T_{g_1}(U,V) = \left(\log\left(\frac{1+U+V}{U+V}\right), \log\left(\frac{1+U}{U}\right) - \log\left(\frac{1+U+V}{U+V}\right)\right). \tag{2.18}$$

As shows (2.18) it is useful to introduce

$$(U',V') = \left(\frac{1+\frac{1}{U+V}}{1+\frac{1}{U}}, U+V\right). \tag{2.19}$$

Obviously the correspondence $(U,V) \mapsto (U',V')$ is one-to one:

$$(U,V) = \left(\frac{U'V'}{V'+1-U'V'}, \frac{V'(V'+1)(1-U')}{V'+1-U'V'}\right). \tag{2.20}$$

Furthermore, (X,Y) can be easily expressed in terms of (U',V'):

$$X=\log(1+1/V') \text{ and } Y=-\log U'. \tag{2.21}$$

Since it is easy to determine the density function of $\phi(\xi)$ knowing the density function of a r.v. ξ, where ϕ is differentiable and bijective, then Theorem 2.4 and its analogue related to $f=g_1$ (cf Remark 2.5) are equivalent to Theorem 2.6 below.

Theorem 2.6 a) *Let U' and V' be two positive and independent random variables. The r.v.'s U and V defined by (2.20) are independent if only if there exist some constants a, b, c such that*

$$U'\sim \mathrm{Beta}(a,b) \text{ and } V'\sim K^{(2)}(a+b,-b,c). \tag{2.22}$$

If one of these equivalent conditions holds, then $U\sim K^{(2)}(a,b,p)$ and $V\sim\gamma(b,c)$.

b) *Let U and V be two positive and independent random variables. The r.v.'s U' and V' defined by (2.19) are independent if only if there exist some constants a, b, c such that*

$$U\sim K^{(2)}(a,b,c) \text{ and } V\sim\gamma(b,c). \tag{2.23}$$

Under (2.23), $U'\sim \mathrm{Beta}(a,b)$ and $V'\sim K^{(2)}(a+b,-b,c)$.

Let us formulate an easy consequence of Theorem 2.6.

Theorem 2.7 *For any $a,b,c>0$, the transformation $(u,v)\mapsto(\frac{1+\frac{1}{u+v}}{1+\frac{1}{u}},\ u+v)$ maps the probability measure $K^{(2)}(a,b,c)\otimes\gamma(b,c)$ to the probability measure $\mathrm{Beta}(a,b)\otimes K^{(2)}(a+b,-b,c)$. In particular:*

$$K^{(2)}(a,b,c)*\gamma(b,c)=K^{(2)}(a+b,-b,c). \tag{2.24}$$

Remark 2.8 Note that (2.24) may be regarded as an analogue of (1.2).

2.2 The case $f=f^*_\delta$

Recall that f^*_δ has been defined by (2.13). Due to the form of f^*_δ, a change of variables allows to simplify the search of independent r.v.'s X and Y such that the two components of $T_{f^*_\delta}(X,Y)$ are independent.

For any decreasing and bijective function $f:(0,\infty)\to(0,\infty)$ we define

$$\overline{f}(x)=\exp\{-f(-\log x)\}, \quad x\in(0,1), \tag{2.25}$$

$$T^m_f(x,y)=\left(f(xy),\frac{f(x)}{f(xy)}\right), \quad x,\,y\in(0,1). \tag{2.26}$$

Observe that $\overline{f}$ is one-to-one from $(0,1)$ onto $(0,1)$, T^m_f is one-to-one from $(0,1)^2$ onto $(0,1)^2$ and

$$\left(T^m_f\right)^{-1}=T^m_{f^{-1}}. \tag{2.27}$$

Definition 2.9 *Let X and Y be two independent and $(0,1)$-valued random variables. We say that a decreasing and bijective function $f:(0,1)\to(0,1)$ is a multiplicative LWMY function with respect to (X,Y) if the r.v.'s $U^m := f(XY)$ and $V^m := \frac{f(X)}{f(XY)}$ are independent.*

Remark 2.10 *For any random vector (X,Y) in $(0,\infty)^2$ we consider $X' = e^{-X}$, $Y' = e^{-Y}$. Then f is a LWMY function with respect to (X,Y) if and only if $\overline{f}$ is a multiplicative LWMY function with respect to (X',Y').*

The change of variable $x' = e^{-x}$ is very convenient since the function

$$\phi_\delta(x) := \overline{f_\delta^*}(x) = \frac{1-x}{1+(\delta-1)x}, \quad x\in(0,1) \tag{2.28}$$

is homographic.
Note that $\overline{f_\delta^*}:(0,1)\to(0,1)$ is bijective, decreasing and equal to its inverse.
First, let us determine the distribution of the couple (X',Y') of r.v.'s such that ϕ_δ is a multiplicative LWMY function with respect to (X',Y').

For $a,b,\alpha>0$ and $c\in\mathbb{R}$ consider the probability measure

$$\beta_\alpha(a,b;c)(dx) = k_\alpha(a,b;c)x^{a-1}(1-x)^{b-1}(\alpha x+1-x)^c\mathbf{1}_{(0,1)}(x)dx. \tag{2.29}$$

Note that if $c=0$, then $\beta_\alpha(a,b;c) = \mathrm{Beta}(a,b)$.

Theorem 2.11 *Let X' and Y' be two independent random variables valued in $(0,1)$. Consider*

$$(U^m,V^m) = T^m_{\phi_\delta}(X',Y') = \left(\frac{1-X'Y'}{1+(\delta-1)X'Y'}, \frac{1-X'}{1+(\delta-1)X'}\frac{1+(\delta-1)X'Y'}{1-X'Y'}\right)$$

for fixed $\delta>0$.
Then, U^m and V^m are independent if and only if there exist $a,b,\lambda>0$ such that

$$X' \sim \beta_\delta(a+b,\lambda;-\lambda-b),\ Y'\sim\mathrm{Beta}(a,b). \tag{2.30}$$

If this condition holds, then

$$U^m \sim \beta_\delta(\lambda+b,a;-a-b),\ V^m\sim\mathrm{Beta}(\lambda,b). \tag{2.31}$$

In the case $\delta=1$, Theorem 2.11 takes a very simple form.

Proposition 2.12 *Let X' and Y' be two independent random variables valued in $(0,1)$. Then*

$$U^m = 1-X'Y',\ V^m = \frac{1-X'}{1-X'Y'}$$

are independent if and only if there exist $a,b,\lambda>0$ such that

$$X'\sim\mathrm{Beta}(a+b,\lambda) \text{ and } Y'\sim\mathrm{Beta}(a,b).$$

If one of these conditions holds, then $U^m\sim\mathrm{Beta}(\lambda+b,a)$ and $V^m\sim\mathrm{Beta}(\lambda,b)$.

Remark 2.13 When $X' \sim \mathrm{Beta}(a+b,\lambda)$ and $Y' \sim \mathrm{Beta}(a,b)$ it can be proved that U^m and V^m are independent using the well-known property: if Z and Z' are independent, $Z \sim \gamma(a,1)$ and $Z' \sim \gamma(b,1)$ then $R := \frac{Z}{Z+Z'}$ and $Z+Z'$ are independent and $R \sim \mathrm{Beta}(a,b)$ and $Z+Z' \sim \gamma(a+b,1)$ (see for instance Yor, 1989).

According to Remark 2.10, f_δ^* is a LWMY function with respect to (X,Y) if and only if ϕ_δ is a multiplicative LWMY function with respect to $(X',Y') = (e^{-X}, e^{-Y})$. Therefore, a classical change of variables allows to deduce that Theorem 2.11 is equivalent to Theorem 2.14 below:

Theorem 2.14 *1) Consider two positive and independent random variables X and Y. The random variables $U = f_\delta^*(X+Y)$, $V = f_\delta^*(X) - f_\delta^*(X+Y)$ are independent if and only if the densities of Y and X are respectively*

$$\begin{aligned} p_Y(y) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}} && (2.32)\\ p_X(x) &= k_\delta(a+b,\lambda,-\lambda-b)e^{-(a+b)x}(\delta e^{-x}+1-e^{-x})^{-\lambda-b} \\ &\quad \times(1-e^{-x})^{\lambda-1}\mathbf{1}_{x>0} && (2.33)\end{aligned}$$

where $a,b>0$, $\lambda \in \mathbb{R}$ and $k_\delta(a+b,\lambda,-\lambda-b)$ is the normalizing factor (cf (2.29)). Thus e^{-Y} is Beta(a,b) distributed and e^{-X} is $\beta_\delta(a+b,\lambda,-\lambda-b)$ distributed.

2) If 1. holds then the densities of U and V are respectively

$$\begin{aligned} p_U(u) &= k_\delta(\lambda+b,a;-a-b)e^{-u(\lambda+b)}(1-e^{-u})^{a-1} \\ &\quad \times(1+(\delta-1)e^{-u})^{-a-b}\mathbf{1}_{u>0}, && (2.34)\\ p_V(v) &= e^{-\lambda v}(1-e^{-v})^{b-1}\,\mathbf{1}_{v>0}. && (2.35)\end{aligned}$$

We skip the proof of Theorem 2.14 since it is similar to that of Theorem 2.4.

3 The set of all possible "smooth" LWMY functions

The following theorem gives a functional equation linking LWMY functions to the related densities.

Theorem 3.1 *Let X and Y be two independent and positive random variables whose densities p_X and p_Y are positive and twice differentiable. Define $\phi_X = \log p_X$ and $\phi_Y = \log p_Y$. Consider a decreasing function $f : (0,\infty) \mapsto (0,\infty)$, three times differentiable. Then f is a LWMY function with respect to (X,Y) if and only if*

$$\begin{aligned} &\phi_X''(x) - \phi_X'(x)\frac{f''(x)}{f'(x)} + \phi_Y''(y)f'(x)\left(\frac{1}{f'(x)} - \frac{1}{f'(x+y)}\right) \\ &+\phi_Y'(y)\frac{f''(x)}{f'(x)} + \frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = 0, \quad x,y>0. \qquad (3.1)\end{aligned}$$

Proof : Let $g = f^{-1}$ and $(U, V) = T_f(X, Y)$. By formula (2.9), $(X, Y) = T_g(U, V)$. X and Y being independent, the density of (U, V) is

$$p_{(U,V)}(u, v) = p_X(g(u+v))\, p_Y(g(u) - g(u+v))\, |J(u, v)| 1_{u,v>0} \tag{3.2}$$

where J is the Jacobian of the transformation T_f. One gets $|J(u, v)| = g'(u+v)g'(u)$, then

$$p_{(U,V)}(u, v) = p_X(g(u+v))\, p_Y(g(u) - g(u+v))\, g'(u+v)g'(u). \tag{3.3}$$

The variables U and V are independent if and only if the function $H = \log p_{(U,V)}$ satisfies $\frac{\partial^2 H}{\partial u \partial v} = 0$. By Equation (3.3) we obtain

$$\begin{aligned} \frac{\partial^2 H}{\partial u \partial v} &= \phi''_X(x)[g'(f(x))]^2 + \phi'_X(x) g''(f(x)) \\ &\quad - \phi''_Y(y) g'(f(x)) \left[g'(f(x+y)) - g'(f(x)) \right] \\ &\quad - \phi'_Y(y) g''(f(x)) + \frac{g'''g' - (g'')^2}{(g')^2}(f(x)) \end{aligned} \tag{3.4}$$

where $x = g(u+v)$ and $y = g(u) - g(u+v)$. Differentiating three times the relation $g(f(x)) = x$, we obtain $g''(f(x)) = -\frac{f''(x)}{f'(x)^3}$ and $g'''(f(x)) = -\frac{f'''(x)f'(x) - 3f''(x)^2}{f'(x)^5}$. As a result,

$$\frac{g'''g' - (g'')^2}{(g')^2}(f(x)) = \frac{2f''(x)^2 - f'''(x)f'(x)}{f'(x)^4}. \tag{3.5}$$

Therefore, $\frac{\partial^2 H}{\partial u \partial v} = 0$ leads to (3.1). □

We restrict ourselves to *smooth* LWMY functions f, i.e. satisfying

$$f : (0, \infty) \to (0, \infty) \text{ is bijective and decreasing,} \tag{3.6}$$

$$f \text{ is three times differentiable,} \tag{3.7}$$

$$F(x) = \sum_{n \geq 1} a_n x^n, \ \forall x > 0. \tag{3.8}$$

where $F := 1/f'$.
According to (3.6), $f'(0_+) = -\infty$. This implies $F(0_+) = 0$ and explains why the series in (3.8) starts with $n = 1$.

The goal of this section is to prove half of Theorem 2.2: if f is a smooth LWMY function, then f belongs to one of the four classes $\mathcal{F}_1, \ldots, \mathcal{F}_4$ introduced in Remark 2.3. First, we characterize in Theorem 3.2 all possible functions F. Second, we determine the associated functions f (see Theorem 3.9).

Theorem 3.2 *Suppose that f is a smooth LWMY function and that the assumptions of Theorem 3.1 are satisfied.*

1. *If $F'(0_+) = 0$, then $a_2 < 0$ and*

$$F(x) = \begin{cases} \frac{a_2^2}{6a_4} \left(\cosh\left(x \sqrt{\frac{12a_4}{a_2}} \right) - 1 \right) & \text{if } a_4 < 0 \\ a_2 x^2 & \text{otherwise.} \end{cases} \tag{3.9}$$

2. *If $F'(0_+) \neq 0$, then*

$$F(x) = \begin{cases} \frac{a_1 a_2}{3a_3}\left[\cosh\left(x\sqrt{\frac{6a_3}{a_1}}\right) - 1\right] + a_1\sqrt{\frac{a_1}{6a_3}}\sinh\left(x\sqrt{\frac{6a_3}{a_1}}\right) & \text{if } a_1 a_3 > 0 \\ a_1 x + a_2 x^2 & \text{otherwise.} \end{cases} \tag{3.10}$$

Remark 3.3 *Unsurprisingly, the case $F(x) = a_2 x^2$ corresponds to $f(x) = -\frac{1}{a_2}\frac{1}{x}$, i.e. the case considered by Matsumoto-Yor and Letac-Wesolowski.*

Throughout this subsection we suppose that f satisfies (3.6)-(3.8) and that the assumptions of Theorem 3.1 are fulfilled. For simplicity of statement of results below we do not repeat these conditions.

Recall that ϕ_Y is the logarithm of the density of Y. Let us introduce

$$h := \phi_Y' \tag{3.11}$$

Lemma 3.4 1. *There exists a function $\lambda : (0, \infty) \to \mathbb{R}$ such that*

$$F(x+y) = \frac{\lambda(x) - h(y)F'(x)}{h'(y)} + F(x). \tag{3.12}$$

2. *F satisfies*

$$F(y) = \frac{\lambda(0_+) - h(y)F'(0_+)}{h'(y)}. \tag{3.13}$$

Remark 3.5 *Suppose that we have been able to determine F. Then, $h = \phi_Y'$ solves the linear ordinary differential equation (3.13) and can therefore be determined. The remaining function ϕ_X is obtained by solving Equation (3.1).*

Proof of Lemma 3.4 :
Using (3.11) and $F = 1/f'$ in Equation (3.1), we obtain

$$c(x) = h(y)\frac{F'(x)}{F(x)} + h'(y)\frac{1}{F(x)}(F(x+y) - F(x))$$

where $c(x)$ depends only on x. Multiplying both sides by $F(x)$ and taking the y-derivative leads to

$$0 = F'(x)h'(y) + [F(x+y) - F(x)]h''(y) + h'(y)F'(x+y).$$

Fix $x > 0$. Then $\theta(y) := F(x+y)$ is a solution of the differential equation in y:

$$0 = F'(x)h'(y) + (\theta(y) - F(x))h''(y) + h'(y)\theta'(y). \tag{3.14}$$

A solution of the related homogeneous equation in y is $\frac{\rho}{h'(y)}$ where ρ is a constant. It is easy to prove that $y \mapsto -F'(x)h(y) + F(x)h'(y)$ solves (3.14). Thus, the general solution of (3.14) is

$$\theta(y) = \frac{1}{h'(y)}\left[\lambda(x) - F'(x)h(y) + F(x)h'(y)\right].$$

Since $\theta(y) = F(x+y)$, (3.12) follows.

According to (3.8), $F(0_+)$ and $F'(0_+)$ exist. Therefore, taking the limit $x \to 0_+$ in (3.12) implies both the existence of $\lambda(0_+)$ and relation (3.13). $\square$

The following lemma shows that the function F (and thus f) solves a self-contained equation in which h, and thereby the densities of X and Y, are not involved.

Lemma 3.6 *F solves the delay equations :*

$$F(x+y) = \frac{F(y)[\lambda(x) - h(y)F'(x)]}{\lambda(0_+) - h(y)F'(0_+)} + F(x) \qquad (x, y > 0) \tag{3.15}$$

$$F'(x+y) = \frac{F'(y) + F'(0_+)}{F(y)}[F(x+y) - F(x)] - F'(x) \qquad (x, y > 0). \tag{3.16}$$

Proof:
By (3.13) we have

$$h'(y) = \frac{\lambda(0_+) - h(y)F'(0_+)}{F(y)}.$$

Equation (3.15) then follows by rewriting Equation (3.12) and replacing $h'(y)$ with the expression above.
We differentiate (3.15) in y and use the fact that $\lambda(0_+) - h(y)F'(0_+) = h'(y)F(y)$ to obtain :

$$F'(x+y) = [F'(y) + F'(0_+)]\frac{\lambda(x) - h(y)F'(x)}{F(y)h'(y)} - F'(x).$$

By (3.12) we have $\frac{\lambda(x)-h(y)F'(x)}{F(y)h'(y)} = \frac{F(x+y)-F(x)}{F(y)}$ and this gives (3.16). $\square$

Remark 3.7 We can see (3.16) as a scalar neutral delay differential equation. Indeed, set $t = x+y$ and consider $y > 0$ as a fixed parameter. Then (3.16) becomes:

$$F'(t) = a(F(t) - F(t-y)) - F'(t-y), \qquad t \geq y, \tag{3.17}$$

where $a := \frac{F'(y)+F'(0_+)}{F(y)}$. Replacing $F(t)$ in (3.17) with $e^{at}G(t)$ leads to:

$$G'(t) + e^{-ay}G'(t-y) + 2ae^{-ay}G(t-y) = 0, \qquad t \geq y. \tag{3.18}$$

Equation (3.18) is called a neutral delay differential equation (cf for instance, Section 6.1, in Györi and Ladas, 1991). These equations have been intensively studied but the authors only focused on the asymptotic behaviour of the solution as $t \to \infty$. Unfortunately, these results give no help to solve explicitly either (3.16) or (3.18).

Lemma 3.8 *For all integers $k \geq 0$ and $l \geq 1$, we have*

$$\sum_{m=0}^{l-1}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = (l-2)(k+1)a_{k+1}a_l + a_1a_{l+k}C^k_{l+k}, \tag{3.19}$$

$$C^k_{k+3}a_{k+3}a_1 = (k+1)a_{k+1}a_3, \tag{3.20}$$

$$2C^k_{k+4}a_{k+4}a_1 + C^k_{k+3}a_{k+3}a_2 - C^k_{k+2}a_{k+2}a_3 - 2(k+1)a_{k+1}a_4 = 0, \tag{3.21}$$

where $C^p_n = \frac{n!}{(n-p)!p!}$.

Proof: Obviously Equation (3.16) is equivalent to:

$$F'(x+y)F(y) = F'(y)F(x+y) - F'(y)F(x) - F(y)F'(x) + F'(0_+)F(x+y) - F'(0_+)F(x). \tag{3.22}$$

Using the asymptotic expansion (3.8) of F we can develop each term in (3.22) as a series with respect to x and y. Then, identifying the series on the right-hand side and the left-hand side we get (3.19)-(3.21). The details are provided in Appendix.

□

Proof of Theorem 3.2 We only prove item 1. The proof of item 2. is similar.

Since $a_1 = F'(0_+) = 0$, we necessarily have $a_2 \neq 0$. Indeed, if $a_2 = 0$ then, by (3.21) with $k = 1$, we would have $-3a_3^2 - 4a_2a_4 = 0$, i.e. $a_3 = 0$, and using again (3.21) with $k = 3$ would imply $a_4 = 0$ and finally $a_k = 0$ for every $k \geq 0$, which is a contradiction because, by definition, $F = 1/f'$ does not vanish.

So, we have $a_1 = 0$ and $a_2 \neq 0$. Equation (3.20) with $k = 1$ reads $4a_4a_1 = 2a_2a_3$, which implies $a_3 = 0$. Applying (3.20) to $k = 2n$ provides, by induction on n, $a_{2n+1} = 0$ for every $n \geq 0$.

Therefore, Equation (3.21) reduces to $(k+3)(k+2)(k+1)a_{k+3}a_2 = 12(k+1)a_{k+1}a_4$, $(k \geq 0)$. i.e. $a_{k+3} = \frac{12a_4}{a_2}\frac{1}{(k+3)(k+2)}a_{k+1}$. This leads to

$$a_{2k} = \left(\frac{12a_4}{a_2}\right)^{k-1}\frac{2}{(2k)!}a_2, \qquad k \geq 1. \tag{3.23}$$

Then, $F(x) = a_2x^2$ if $a_4 = 0$ and if $a_4 \neq 0$ we have

$$F(x) = \sum_{k\geq 1}\left(\frac{12a_4}{a_2}\right)^{k-1}\frac{2}{(2k)!}a_2x^{2k}.$$

If $a_4a_2 < 0$, then $F(x) = \frac{a_2^2}{6a_4}\left[\cos\left(x\sqrt{\frac{-12a_4}{a_2}}\right) - 1\right]$. This implies $F(2\pi\sqrt{\frac{-12a_4}{a_2}}) = 0$ which is impossible since $F(x) = 1/f'(x) < 0$. Consequently,

$$F(x) = \frac{a_2^2}{6a_4}\left[\cosh\left(x\sqrt{\frac{12a_4}{a_2}}\right) - 1\right]. \qquad \square$$

Now, in each case of Theorem 3.2 we compute the function f associated with F via the relation $F = 1/f'$. We do not detail the calculations since they reduce to get a good primitive of $1/F$. Recall that we restrict ourselves to functions f satisfying (3.6)-(3.8) and work under the assumptions of Theorem 3.1.

Theorem 3.9 1. *If $F(x) = a_2x^2$ then $f(x) = \frac{1}{a_2x}$.*

2. *If $F(x) = \alpha(\cosh\beta x - 1), \alpha, \beta > 0$, then $f(x) = \frac{2}{\alpha\beta}f_1(\beta x)$.*

3. *If $F(x) = a_1x + a_2x^2$ then $f(x) = -\frac{1}{a_1}g_1(\frac{a_2}{a_1}x)$.*

4. *If* $F(x) = \frac{a_1 a_2}{3a_3}\left[\cosh\left(x\sqrt{\frac{6a_3}{a_1}}\right) - 1\right] + a_1\sqrt{\frac{a_1}{6a_3}}\sinh\left(x\sqrt{\frac{6a_3}{a_1}}\right)$ *then*
$$f(x) = -\frac{1}{\beta\gamma}\log\left(\frac{e^{\beta x}+\delta-1}{e^{\beta x}-1}\right),$$
where $\alpha = \frac{a_1a_2}{3a_3}$, $\beta = \sqrt{\frac{6a_3}{a_1}}$ *and* $\gamma = a_1\sqrt{\frac{a_1}{6a_3}}$.

□

4 Proof of Theorem 2.4

Recall that $\phi_Y = \log p_Y$, $h = \phi_Y'$ and $F'(0_+) = 0$. It is easy to deduce from (3.13) that there exist constants λ and c_1 such that $h(y) = \lambda f(y) + c_1$, i.e. $h(y) = \frac{\lambda e^y}{e^y-1} + c_1 - \lambda$. This implies the existence of a constant d such that $\phi_Y(y) = \lambda\log(e^y-1) + (c_1-\lambda)y + d$. Setting $M = e^d$, we have by integration, for all $y > 0$,

$$p_Y(y) = M(1-e^{-y})^\lambda e^{c_1 y}. \tag{4.1}$$

To give more information on the normalizing constant M, one observes, for $a = -c_1$ and $b = \lambda + 1$, that

$$\int_0^\infty M(1-e^{-y})^{b-1}e^{-ay}\,dy = M\int_0^1 (1-u)^{b-1}u^{a-1}\,du$$

which implies that $a > 0$, $b > 0$ and $M = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$. This proves (2.16).

To find the density of X we come back to Equation (3.1) and compute each of its terms.

We have $f'(x) = \frac{-e^x}{(e^x-1)^2}$, $f''(x) = \frac{e^{2x}+e^x}{(e^x-1)^3}$, $f'''(x) = -\frac{e^{3x}+4e^{2x}+e^x}{(e^x-1)^4}$, so that $\frac{f'(x)}{f'(x+y)} = \frac{e^{-y}(e^{x+y}-1)^2}{(e^x-1)^2}$ and $\frac{f''(x)}{f'(x)} = -\frac{e^x+1}{e^x-1}$. Calculations yield

$$\frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = \frac{e^{2x}+1}{(e^x-1)^2}. \tag{4.2}$$

Moreover,

$$-\phi_Y'(y)\frac{f''(x)}{f'(x)} + \phi_Y''(y)\left(\frac{f'(x)}{f'(x+y)} - 1\right) = \frac{(c_1-\lambda)e^{2x} - c_1}{(e^x-1)^2}. \tag{4.3}$$

Equation (3.1) can then be writen, using (4.2) and (4.3):

$$\phi_X''(x) + \frac{e^x+1}{e^x-1}\phi_X'(x) = \frac{(c_1-\lambda-1)e^{2x} - c_1 - 1}{(e^x-1)^2}.$$

Then $h_0 := \phi_X'$ solves

$$h_0'(x) + \frac{e^x+1}{e^x-1}h_0(x) = \frac{(c_1-\lambda-1)e^{2x} - c_1 - 1}{(e^x-1)^2}. \tag{4.4}$$

Note that $x \mapsto \frac{K}{4\sinh^2(x/2)}$ solves (4.4) with the right-hand side equal to 0, and $x \mapsto \frac{(c_1-\lambda-1)e^x+(c_1+1)e^{-x}}{4\sinh^2(x/2)}$ is a particular solution of (4.4). Therefore, the solution of (4.4) is

$$h(x) = \frac{(c_1-\lambda-1)e^x+(c_1+1)e^{-x}+K}{4\sinh^2(\frac{x}{2})}$$

for some constant K. This implies

$$\phi'_X(x) = c_1+1+\frac{(2c_1-\lambda+K)e^x}{(e^x-1)^2}-\frac{(\lambda+2)e^x}{e^x-1}.$$

As a consequence, there exists a constant δ such that

$$\phi_X(x) = (c_1+1)x-\frac{(2c_1-\lambda+K)e^x}{e^x-1}-(\lambda+2)\log(e^x-1)+\delta.$$

Thus $p_X(x) = Ne^{(c_1+1)x}(e^x-1)^{-\lambda-2}\exp\left(-\frac{2c_1-\lambda+K}{e^x-1}\right)\mathbf{1}_{\{x>0\}}$. Recall that $a=-c_1$ and $b=\lambda+1$. With $c=2c_1-\lambda+K$ one gets (2.17). More information on the constant N is obtained by observing that if we set $V'=f_1(X)=\frac{1}{e^X-1}$, then the density of V' is

$$f_{V'}(w) = N(w+1)^{-a}w^{a+b-1}\exp\{-cw\}\mathbf{1}_{\{w>0\}},$$

i.e. the law of V' is $K^{(2)}(a+b,-b,c)$ (cf Equation (2.14)).

We have $g'_1(u) = -\frac{1}{u(u+1)}$. A computation of Jacobian, together with (2.16) and (2.17), imply, for $u,v>0$,

$$\begin{aligned} p_{(U,V)}(u,v) &= p_X\left(\log\left[\frac{u+v+1}{u+v}\right]\right)p_Y\left(\log\left[\frac{(u+1)(u+v)}{u(u+v+1)}\right]\right) \\ &\quad\times\frac{1}{u(u+1)(u+v)(u+v+1)}. \end{aligned}$$

Then we get that $p_{(U,V)}(u,v)$ is the product of a function of u and a function of v and this gives item 2. of Theorem 2.4. $\square$

5 Appendix

5.1 Proof of Lemma 3.8

We have

$$F'(x+y)F(y) = \sum_{k\geq 0}x^k\sum_{m\geq 0,n\geq 1+k}na_na_mC^k_{n-1}y^{n+m-1-k}.$$

Setting $l=m+n-1-k$ for fixed m gives

$$F'(x+y)F(y) = \sum_{k\geq 0,l\geq 0}x^ky^l\sum_{m=0}^{l}(l-m+1+k)C^k_{l-m+k}a_{l-m+1+k}a_m. \tag{5.5}$$

By the same method we have

$$F'(y)F(x+y) = \sum_{k\geq 0, l\geq 0} x^k y^l \left(\sum_{m=0}^{l+1} m C^k_{l-m+k+1} a_{l-m+1+k} a_m \right). \tag{5.6}$$

As for the two other terms of (3.22) we get

$$\begin{aligned} F'(y)F(x) &= \sum_{k\geq 0, l\geq 0} a_k a_{l+1}(l+1) x^k y^l & (5.7)\\ F'(x)F(y) &= \sum_{k\geq 0, l\geq 0} a_{k+1} a_l (k+1) x^k y^l. & (5.8) \end{aligned}$$

Consequently,

$$F'(0_+)F(x+y) = a_1 \sum_{n\geq 0} a_n (x+y)^n = a_1 \sum_{k,l\geq 0} a_{l+k} C^k_{l+k} x^k y^l, \tag{5.9}$$

$$F'(0_+)F(x) = a_1 \sum_{k\geq 0} a_k x^k. \tag{5.10}$$

Identifying the coefficient of $x^k y^l$ in (3.22) and using (5.5) to (5.10) we have, for $k \geq 0$ and $l \geq 0$:

$$\begin{aligned} \sum_{m=0}^{l} (l-m+1+k) C^k_{l-m+k} a_{l-m+1+k} a_m &= -(l+1) a_k a_{l+1} - (k+1) a_{k+1} a_l \\ &\quad + \sum_{m=0}^{l+1} m C^k_{l-m+k+1} a_{l-m+1+k} a_m \\ &\quad + a_1 a_{l+k} C^k_{l+k} - a_1 a_k 1_{l=0}. \end{aligned} \tag{5.11}$$

Note that if $l = 0$, both sides of (5.11) vanish, therefore we may suppose in the sequel that $l \geq 1$.
For $m = l+1$ we have $m C^k_{l-m+k+1} a_{l-m+1+k} a_m = (l+1) a_k a_{l+1}$. Thus, Equation (5.11) reads

$$\begin{aligned} \sum_{m=0}^{l} (l-m+1+k) C^k_{l-m+k} a_{l-m+1+k} a_m &= -(k+1) a_{k+1} a_l + \sum_{m=0}^{l} m C^k_{l-m+k+1} a_{l-m+1+k} a_m \\ &\quad + a_1 a_{l+k} C^k_{l+k}. \end{aligned} \tag{5.12}$$

But one finds by a calculation using the definition that

$$(l-m+1+k) C^k_{l-m+k} - m C^k_{l-m+1+k} = (l-2m+1) C^k_{l-m+1+k},$$

so that Equation (5.12) is equivalent to

$$\sum_{m=0}^{l} (l-2m+1) C^k_{l-m+1+k} a_{l-m+1+k} a_m = -(k+1) a_{k+1} a_l + a_1 a_{l+k} C^k_{l+k}. \tag{5.13}$$

For $m = l$ we have $(l - 2m + 1)C^k_{l-m+1+k} a_{l-m+1+k} a_m = (1 - l)(k + 1)a_{k+1}a_l$. Consequently Equation (5.13) may be written as follows:

$$\sum_{m=0}^{l-1} (l - 2m + 1)C^k_{l-m+1+k} a_{l-m+1+k} a_m - (l - 1)(k + 1)a_{k+1}a_l = -(k + 1)a_{k+1}a_l + a_1 a_{l+k} C^l_{l+k}$$

which implies (3.19).
(3.20) and (3.21) follow by applying (3.19) to $l = 3$ and $l = 4$ respectively.

□

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References

[1] Barndorff-Nielsen, O. E. and Halgreen, C. (1977). Infinite divisibility of the Hyperbolic and generalized inverse Gaussian distribution. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **38**, 309-312.

[2] Barndorff-Nielsen, O. E. and Koudou, A. E. (1998). Trees with random conductivities and the (reciprocal) inverse Gaussian distribution. *Adv. Appl. Probab.* **30**, 409-424.

[3] Bhattacharya, R. N. and Waymire, E. C. (1990). *Stochastic Processes with Applications.* New York: Wiley.

[4] Dyson, F. J. (1953). The dynamics of a disordered linear chain. *Phys. Rev.* **92**, 1331-1338.

[5] Good, I. J. (1953).The population frequencies of species and the estimation of population parameters, *Biometrika*, **40**, 237-260.

[6] Györi, I. and Ladas, G., *Oscillation theory of delay differential equations* (Oxford Science Publications, 1991).

[7] Koudou, A. E. (2006). A link between the Matsumoto-Yor property and an independence property on trees. *Stat. and Probab. Letters* **76**, 1097-1101.

[8] Koudou, A. E. and Vallois, P. (2010). Another proof of the characterization of GIG distributions by the Matsumoto-Yor property. *Preprint.*

[9] Letac, G. and Seshadri, V. (1983). A characterization of the generalized inverse Gaussian distribution by continued fractions. *Z. Wahr. verw. Geb.* **62**, 485-489.

[10] Letac, G. and Wesolowski, J. (2000). An independence property for the product of GIG and gamma laws. *Ann. Prob.* **28**, 1371-1383.

[11] Marklov, J., Tourigny, Y. and Wolowski, L. (2008). Explicit invariant measures for products of random matrices. *Trans. Amer. Soc.* **360**, 3391-3427.

[12] Matsumoto, H. and Yor, M. (2001). An analogue of Pitman's $2M - X$ theorem for exponential Wiener functional, Part II: the role of the generalized inverse Gaussian laws. *Nagoya Math. J.* **162**, 65-86.

[13] Matsumoto, H. and Yor, M. (2003). Interpretation via Brownian motion of some independence properties between GIG and gamma variables. *Stat. and Probab. Letters* **61**, 253-259.

[14] Ng, K. W. and Kotz, S. (1995). Kummer-Gamma and Kummer-Beta univariate and multivariate distributions. *Research report*, Department of Statistics, The University of Hong Kong, Hong Kong.

[15] Nagar and Gupta (2002). Matrix-variate Kummer-beta distributions. *J. Austral. Math. Soc.* **73**, 11-25.

[16] Vallois, P. (1989). Sur le passage de certaines marches aléatoires planes au-dessus d'une hyperbole quilatère. *Ann. Inst. Henri poincaré*, **25**, 443-456.

[17] Vallois, P. (1991). La loi Gaussienne inverse généralisée comme premier ou dernier temps de passage de diffusion. *Bull. Sc. Math.*, 2^e Série, **115**, 301-368.

[18] Yor, M. (1989). Une extension markovienne de l'algèbre des lois béta-gamma. *C. R. Acad. Sci. Paris Sér. I. Math.* **308**, 257-260.